Some Sufficient Conditions for the Controllability of Wave Equations with Variable Coefficients

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Abstract

In this work, we present some easily verifiable sufficient conditions that guarantee the controllability of wave equations with non-constant coefficients. These conditions work as complements for those obtained in [3].

1 Introduction and the Main Results

Let $T > 0$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain with a C^2 boundary $\partial\Omega$. Let $a^{ij} \in C^1(\overline{\Omega})(i, j = 1, \cdots, n)$ such that $a^{ij} = a^{ji}$ and $A \overset{\triangle}{=} (a^{ij})_{1\le i,j\le n}$ is a uniformly positive definite matrix. Consider the following hyperbolic equation:

$$\begin{cases} y_{tt} - \displaystyle\sum_{i,j=1}^{n} \left(a^{ij} y_{x_i}\right)_{x_j} = 0 & \text{in } (0,T)\times\Omega, \\ y = 0 & \text{on } (0,T)\times\partial\Omega, \\ y(0) = y_0,\ y_t(0) = y_1 & \text{on } \Omega. \end{cases} \tag{1.1}$$

Here $(y_0, y_1) \in H_0^1(\Omega) \times L^2(\Omega)$. In order to establish the boundary observability estimate for the equation (1.1) by multiplier method or Carleman estimate, one needs the following conditions (see [2] for example):

Condition 1.1. *There exists a function $d \in C^2(\overline{\Omega})$ such that*

$$\sum_{i,j=1}^{n} \left\{ \sum_{i',j'=1}^{n} \left[2a^{ij'} (a^{i'j} d_{x'_i})_{x_{j'}} - a^{ij}_{x_{j'}} a^{i'j'} d_{x_{i'}} \right] \right\} \xi^i \xi^j \ge \mu_0 \sum_{i,j=1}^{n} a^{ij} \xi_i \xi_j, \tag{1.2}$$

when $(x, \xi_1, \cdots, \xi_n) \in \overline{\Omega} \times \mathbb{R}^n$, and such that and

$$|\nabla d| > 0 \quad \text{in } \overline{\Omega}. \tag{1.3}$$

Remark 1.1. *One can directly verify the following: The condition (1.2) is equivalent to that the matrix*

$$B = (b^{ij})_{1\le i,j\le n} \overset{\triangle}{=} \left(\sum_{i',j'=1}^{n} \left(a^{ij'} a^{i'j} d_{x_{i'} x_{j'}} + \frac{a^{ij'} a^{i'j}_{x_{j'}} + a^{jj'} a^{i'i}_{x_{j'}} - a^{ij}_{x_{j'}} a^{i'j'}}{2} d_{x_{i'}} \right) \right)_{1\le i,j\le n} \tag{1.4}$$

is uniformly positive definite.

The function d verifying (1.2) and (1.3) does not exist for some cases. This can be seen from the following example:

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Example 1.1. *Let $\Omega = \{(x, y) : x^2 + y^2 \leq 2\}$. Let $(a^{ij})_{1\leq i,j\leq 2} = \operatorname{diag}(a^1, a^2)$ with $a^1(x, y) = a^2(x, y) = 1 + x^2 + y^2$. By an indirect proof based on the Geometric Control Condition given in [1], we can show that there is no such a function d that satisfies* (1.2).

Now, we study the existence of functions d verifying (1.2) and (1.3) for suitable $(a^{ij})_{1\leq i,j\leq n}$. We will focus our studies on the special case where $A = (a^{ij})_{1\leq i,j\leq n} = \operatorname{diag}(a^1, \cdots, a^n)$, where $a^i \in C^1(\overline{\Omega})$. From Example 1.1, we see that even in this case, the above-mentioned functions d may not exist. Thus, it is interesting to provide certain easily verifiable condition to ensure the existence of such functions d in the case when A is diagonal. The main results of this study are as follows:

Theorem 1.1. *Let $A = \operatorname{diag}(a^1, \cdots, a^n)$, with $a^i \in C^1(\overline{\Omega})$ $(1 \leq i \leq n)$, be positive uniformly definite over $\overline{\Omega}$. If there exists $j \in \{1, \cdots, n\}$ such that all the terms of $\{a^i_{x_j}\}_{1\leq i\leq n; i\neq j}$ remain positive (or negative) over $\overline{\Omega}$, then there is a function $d \in C^2(\overline{\Omega})$ verifying Condition 1.1.*

It is worth mentioning that, in the statement of the main theorem, we don't need the structural condition on $a^j_{x_j}$ where j is the fixed index. Before carrying out the proof, we give two corollaries. The first one corresponds to the case $j = 1$:

Corollary 1.1. *Let $A = \operatorname{diag}(a^1, \cdots, a^n)$, with $a^i \in C^1(\overline{\Omega})$, $i = 1, \cdots, n$, be positive uniformly definite over $\overline{\Omega}$. Suppose that*

$$a^k_{x_1} > 0 \ (\text{or } a^k_{x_1} < 0) \ \text{over } \overline{\Omega}, \ \text{ for } 2 \leq k \leq n, \tag{1.5}$$

then, there is a function $d \in C^2(\overline{\Omega})$ verifying Condition 1.1.

Corollary 1.2. *Let $A = \operatorname{diag}(a^1, a^2)$, with $a^1, a^2 \in C^1(\overline{\Omega})$, be positive uniformly definite over $\overline{\Omega}$. Suppose that $a^1_{x_2}$ (or $a^2_{x_1}$) is either positive or negative over $\overline{\Omega}$. Then there is a function $d \in C^2(\overline{\Omega})$ satisfying Condition 1.1.*

2 Proof of Main Theorem:

Proof of Theorem 1.1: case 1. We first consider the following case:

$$a^i_{x_j} < 0, \quad \text{uniformly over } x \in \overline{\Omega}, \text{ for all } 1 \leq i \leq n \text{ with } i \neq j. \tag{2.1}$$

where j is a fixed index. Let

$$d \triangleq d(x) = e^{\lambda(c+x_j)} + \sum_{1\leq i\leq n, i\neq j} e^{\lambda x_i}, \ \ x \in \overline{\Omega},$$

where $c > 0$ satisfies that

$$\min_{x\in\overline{\Omega}}\{c + x_j\} \geq 1 + \max_{x\in\overline{\Omega}} \sum_{1\leq i\leq n, i\neq j} |x_i|, \tag{2.2}$$

and $\lambda > 0$ is a large number will be determined later. Using (2.2), one could check that the function $d(x)$ enjoys the following properties:

- For any $1 \leq i \leq n$,

$$d_{x_i x_i} > 0, \ d_{x_i} > 0, \qquad \text{uniformly for } x \in \overline{\Omega} \ . \tag{2.3}$$

- For any $1 \leq i \leq n$,

$$\lim_{\lambda\to+\infty} \frac{d_{x_i}}{d_{x_j x_j}} = 0 \qquad \text{uniformly for } x \in \overline{\Omega} \ . \tag{2.4}$$

- For any $1 \leq i \leq n$ with $i \neq j$,

$$\lim_{\lambda\to+\infty} \frac{d_{x_i}}{d_{x_j}} = 0, \quad \lim_{\lambda\to+\infty} \frac{d_{x_i x_i}}{d_{x_j}} = 0, \quad \text{uniformly for } x \in \overline{\Omega} \ . \tag{2.5}$$

From Remark 1.1, to prove d enjoys (1.2) for the case $A = \operatorname{diag}(a^1, \cdots, a^n)$, we only need to show the uniformly positivity of the following matrix:

$$B = \frac{1}{2}\Big(a^i a^j_{x_i} d_{x_j} + a^j a^i_{x_j} d_{x_i}\Big)_{1\le i,j\le n} + \operatorname{diag}\Big((a^1)^2 d_{x_1x_1} - \frac{1}{2}\sum_{k=1}^n a^k a^1_{x_k} d_{x_k}, \cdots, (a^n)^2 d_{x_nx_n} - \frac{1}{2}\sum_{k=1}^n a^k a^n_{x_k} d_{x_k}\Big). \tag{2.6}$$

To achieve this goal, we only need to show that all the leading principal minors of B are positive. In order to avoid the terrible expansion of the determinant, we shall make full use of the asymptotic behavior with respect to the parameter λ. We denote by e_i the i-th standard basis of $\mathbb{R}^n$ and by $\{B_i\}_{i=1}^n$ the row vector of B. It can be verified that, with a very large $\lambda > 0$, the matrix B is uniformly positive definite over $\overline{\Omega}$ if and only if all the leading principal minors of the matrix $\tilde{B}(x,\lambda) := \begin{pmatrix} \frac{B_1}{d_{x_j}} \\ \vdots \\ \frac{B_{j-1}}{d_{x_j}} \\ \frac{B_j}{d_{x_jx_j}} \\ \frac{B_{j+1}}{d_{x_j}} \\ \vdots \\ \frac{B_n}{d_{x_j}} \end{pmatrix}$ is uniformly positive over $\overline{\Omega}$. This later condition is relatively easier to be verified because we could calculate the limit $\tilde{B}(x,+\infty) = \lim_{\lambda\to+\infty} \tilde{B}(x,\lambda)$ and the condition (2.1) guarantees that all the leading principal minors of $\tilde{B}(x,+\infty)$ are uniformly positive over $\overline{\Omega}$. Now we give the details of this:

By (2.6)

$$B_j = \frac{1}{2}\Big(a^j a^l_{x_j} d_{x_l} + a^l a^j_{x_l} d_{x_j}\Big)_{1\le l\le n} + \left((a^j)^2 d_{x_jx_j} - \frac{1}{2}\sum_{k=1}^n a^k a^j_{x_k} d_{x_k}\right) e_j \tag{2.7}$$

Making use of (2.3),(2.4) and (2.5), we deduce

$$\lim_{\lambda\to+\infty} \frac{B_j}{d_{x_jx_j}} = (a^j)^2 e_j \qquad \text{uniformly for } x \in \overline{\Omega}. \tag{2.8}$$

In the same spirit, for $1 \le i \le n$ with $i \ne j$, we have

$$B_i = \frac{1}{2}\Big(a^i a^l_{x_i} d_{x_l} + a^l a^i_{x_l} d_{x_i}\Big)_{1\le l\le n} + \left((a^i)^2 d_{x_ix_i} - \frac{1}{2}\sum_{k=1}^n a^k a^i_{x_k} d_{x_k}\right) e_i \tag{2.9}$$

One could verify by using (2.4) and (2.5) that

$$\lim_{\lambda\to+\infty} \frac{B_i}{d_{x_j}} = \frac{1}{2} a^i a^j_{x_i} e_j - \frac{1}{2} a^j a^i_{x_j} e_i \qquad \text{uniformly for any } x \in \overline{\Omega}. \tag{2.10}$$

By (2.18) and (2.20), we deduce that

$$\lim_{\lambda\to+\infty}\begin{pmatrix} \frac{B_1}{d_{x_j}} \\ \vdots \\ \frac{B_{j-1}}{d_{x_j}} \\ \frac{B_j}{d_{x_jx_j}} \\ \frac{B_{j+1}}{d_{x_j}} \\ \vdots \\ \frac{B_n}{d_{x_j}} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}a^j a^1_{x_j} & \cdots & 0 & \frac{1}{2}a^1 a^j_{x_1} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & -\frac{1}{2}a^j a^{j-1}_{x_j} & \frac{1}{2}a^{j-1}a^j_{x_{j-1}} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & (a^j)^2 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \frac{1}{2}a^{j+1}a^j_{x_{j+1}} & -\frac{1}{2}a^j a^{j+1}_{x_j} & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \frac{1}{2}a^n a^j_{x_n} & 0 & \cdots & -\frac{1}{2}a^j a^n_{x_j} \end{pmatrix} \tag{2.11}$$

uniformly for $x \in \overline{\Omega}$. We deduce from the above formula and (2.3), (2.1) that all the leading principal minors of $\tilde{B}(x,\lambda)$ are uniformly positive with a large λ. This complete the proof. ∎

Proof of Theorem 1.1, case 2. Here we discuss the case when

$$a^i_{x_j} > 0, \quad \text{uniformly over } x \in \overline{\Omega}, 1 \le i \le n, i \ne j, \tag{2.12}$$

where j is a fixed index. In this case, the proof is quite similar as above: we define a function

$$d \triangleq d(x) = e^{-\lambda(x_j - c)} + \sum_{1\le i\le n, i\ne j} e^{-\lambda x_i}, \ \ x \in \overline{\Omega},$$

where $c > 0$ satisfies that

$$\max_{x\in\overline{\Omega}}\{x_j - c\} + 1 \le \min_{x\in\overline{\Omega}} \sum_{1\le i\le n, i\ne j}^{n} |x_i|, \tag{2.13}$$

and $\lambda > 0$ is a large number will be determined later. Using (2.13), one could also check that the function $d(x)$ enjoys the following properties:

- For any $1 \le i \le n$,

$$d_{x_i} < 0, \ d_{x_{ii}} > 0, \qquad \text{uniformly for } x \in \overline{\Omega}\ . \tag{2.14}$$

- For any $1 \le i \le n$,

$$\lim_{\lambda\to+\infty}\frac{d_{x_i}}{d_{x_jx_j}} = 0 \qquad \text{uniformly for } x \in \overline{\Omega}\ . \tag{2.15}$$

- For any $1 \le i \le n$ with $i \ne j$,

$$\lim_{\lambda\to+\infty}\frac{d_{x_i}}{d_{x_j}} = 0, \quad \lim_{\lambda\to+\infty}\frac{d_{x_ix_i}}{d_{x_j}} = 0, \quad \text{uniformly for } x \in \overline{\Omega}\ . \tag{2.16}$$

As before, we deduce from (2.14),(2.15) and (2.16) that the matrix B is uniformly positive definite if and only if all the leading principal minors of the matrix $\hat{B}(x,\lambda) := \begin{pmatrix} -\frac{B_1}{d_{x_j}} \\ \vdots \\ -\frac{B_{j-1}}{d_{x_j}} \\ \frac{B_j}{d_{x_jx_j}} \\ -\frac{B_{j+1}}{d_{x_j}} \\ \vdots \\ -\frac{B_n}{d_{x_j}} \end{pmatrix}$ is

uniformly positive over $\overline{\Omega}$ when λ is large enough. By (2.6)

$$B_j = \frac{1}{2}\Big(a^j a^l_{x_j} d_{x_l} + a^l a^j_{x_l} d_{x_j}\Big)_{1\le l\le n} + \left((a^j)^2 d_{x_j x_j} - \frac{1}{2}\sum_{k=1}^{n} a^k a^j_{x_k} d_{x_k}\right) e_j \tag{2.17}$$

Making use of (2.15) and (2.16), we deduce

$$\lim_{\lambda\to+\infty} \frac{B_j}{d_{x_j x_j}} = (a^j)^2 e_j \qquad \text{uniformly for } x \in \overline{\Omega}. \tag{2.18}$$

In the same spirit, for $1 \le i \le n$ with $i \ne j$, we have

$$B_i = \frac{1}{2}\Big(a^i a^l_{x_i} d_{x_l} + a^l a^i_{x_l} d_{x_i}\Big)_{1\le l\le n} + \left((a^i)^2 d_{x_i x_i} - \frac{1}{2}\sum_{k=1}^{n} a^k a^i_{x_k} d_{x_k}\right) e_i \tag{2.19}$$

One could verify by using (2.4) and (2.5) that

$$\lim_{\lambda\to+\infty} \frac{B_i}{d_{x_j}} = \frac{1}{2} a^i a^j_{x_i} e_j - \frac{1}{2} a^j a^i_{x_j} e_i \qquad \text{uniformly for any } x \in \overline{\Omega}. \tag{2.20}$$

By (2.18) and (2.20), we deduce that

$$\lim_{\lambda\to+\infty}\begin{pmatrix} \frac{B_1}{d_{x_j}} \\ \vdots \\ \frac{B_{j-1}}{d_{x_j}} \\ \frac{B_j}{d_{x_j x_j}} \\ \frac{B_{j+1}}{d_{x_j}} \\ \vdots \\ \frac{B_n}{d_{x_j}} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}a^j a^1_{x_j} & \cdots & 0 & \frac{1}{2}a^1 a^j_{x_1} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & -\frac{1}{2}a^j a^{j-1}_{x_j} & \frac{1}{2}a^{j-1} a^j_{x_{j-1}} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & (a^j)^2 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \frac{1}{2}a^{j+1} a^j_{x_{j+1}} & -\frac{1}{2}a^j a^{j+1}_{x_j} & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \frac{1}{2}a^n a^j_{x_n} & 0 & \cdots & -\frac{1}{2}a^j a^n_{x_j} \end{pmatrix} \tag{2.21}$$

uniformly for $x \in \overline{\Omega}$. The above formula implies

$$\lim_{\lambda\to+\infty}\begin{pmatrix} -\frac{B_1}{d_{x_j}} \\ \vdots \\ -\frac{B_{j-1}}{d_{x_j}} \\ \frac{B_j}{d_{x_j x_j}} \\ -\frac{B_{j+1}}{d_{x_j}} \\ \vdots \\ -\frac{B_n}{d_{x_j}} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}a^j a^1_{x_j} & \cdots & 0 & -\frac{1}{2}a^1 a^j_{x_1} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & \frac{1}{2}a^j a^{j-1}_{x_j} & -\frac{1}{2}a^{j-1} a^j_{x_{j-1}} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & (a^j)^2 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & -\frac{1}{2}a^{j+1} a^j_{x_{j+1}} & \frac{1}{2}a^j a^{j+1}_{x_j} & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & -\frac{1}{2}a^n a^j_{x_n} & 0 & \cdots & \frac{1}{2}a^j a^n_{x_j} \end{pmatrix} \tag{2.22}$$

uniformly for $x \in \overline{\Omega}$. We deduce from the above formula and (2.14), (2.12) that all the leading principal minors of $\hat{B}(x, \lambda)$ are uniformly positive with a large λ. This complete the proof. ∎

3 Examples and Comments

There have been a lot of conditions to ensure the existence of the function d. In [3] (see also [4]), the author provides a sectional curvature condition to guarantee the existence of functions

d. This condition is that the sign of the sectional curvature function k for the Riemannian manifold, with a metric $A^{-1} = (a^{ij})^{-1}_{1\le i,j\le n}$, is either positive or negative over $\overline{\Omega}$.

In this section, we will compare the condition in Theorem 1.1 with the above-mentioned condition given in [3]. Then, we will see some advantage can be taken from the condition in Theorem 1.1. First, [3] needs the C^∞-regularity for coefficients $a^{i,j}$; while our Theorem 1.1 only needs the C^1-regularity for coefficients. Second (more important), there are many cases which can be solved by our Theorem 1.1, but cannot be solved by the sectional curvature condition provided in [3]. Here, we present an example to explain the second advantage above-mentioned.

Example 3.1. *Let $A = \operatorname{diag}(a^1, a^2)$, where $a^1, a^2 \in C^\infty(\overline{\Omega})$. Suppose that $a^2_{x_1} < 0$ over $\overline{\Omega}$. By Theorem 1.1 or Corollary 1.2, there is a function $d \in C^2(\overline{\Omega})$ verifying Condition 1.1. However, by making use of the sectional curvature condition provided in [3], we cannot imply the existence of the above-mentioned d. In fact, after some computation, one can see that the sectional curvature given by the metric A^{-1} is as follows:*

$$k = \frac{1}{4(a^1a^2)^2}\Big[a^2a^1_{x_1}a^2_{x_1} + a^1(a^2_{x_1})^2 - 2a^1a^2a^2_{x_1x_1}\Big]. \tag{3.1}$$

From (3.1), one can construct many such a^i, $i = 1, 2$, with the property that $a^2_{x_1} < 0$ over $\overline{\Omega}$, such that the corresponding k changes its sign over $\overline{\Omega}$.

Here, we provide one of them as follows: Let $\Omega = \{(x_1, x_2) : (x_1 - 2)^2 + x_2^2 < 3/2\} \subset \mathbb{R}^2$. Let $a^1 = e^{\mu_1 x_1}$ and $a^2 = e^{-\mu_2 x_1^2}$, where μ_1 and μ_2 satisfy

$$\mu_1 > 0;\ \mu_2 > 0;\ \mu_1 + 2\mu_2 < 2;\ 3\mu_1 + 18\mu_2 > 2. \tag{3.2}$$

Clearly, (3.2) has solutions.

In this case, it is clear that $a^2_{x_1} < 0$ over $\overline{\Omega}$ because $x_1 > 0$ over $\overline{\Omega}$. From (3.1), we see

$$\begin{aligned} 4(a^1a^2)^2k \quad &= -2\mu_1\mu_2x_1e^{\mu_1x_1-2\mu_2x_1^2} + 4\mu_2^2x_1^2e^{\mu_1x_1-2\mu_2x_1^2} + 4\mu_2e^{\mu_1x_1-2\mu_2x_1^2} - 8\mu_2^2x_1^2e^{\mu_1x_1-2\mu_2x_1^2} \\ &= -2\mu_2e^{\mu_1x_1-2\mu_2x_1^2}\big(\mu_1x_1 + 2\mu_2x_1^2 - 2\big). \end{aligned}$$

From (3.2), it follows that

$$(\mu_1x_1 + 2\mu_2x_1^2 - 2)\big|_{x_1=1} < 0$$

and

$$(\mu_1x_1 + 2\mu_2x_1^2 - 2)\big|_{x_1=3} > 0.$$

Hence, $k > 0$ in the set $\Omega \cap \{(x_1, x_2) : x_1 = 1\}$; while $k < 0$ in the set $\Omega \cap \{(x_1, x_2) : x_1 = 3\}$. From these, we conclude that k changes its sign over $\overline{\Omega}$. Therefore, the method in [3] does not work for the current case.

The next two examples are taken from [3] for which the existence can be ensured by either the sectional curvature condition provided in [3] or our Theorem 1.1.

Example 3.2. *Let $A = (a^{ij})_{1\le i,j\le 2} = \operatorname{diag}(e^{x^3+y^3}, e^{x^3+y^3})$. One can directly check that*

$$a^2_{x_1} = 3y^2e^{x^3+y^3} > 0.$$

Then, according to Theorem 1.2, there is a d satisfiing (1.2) *and* (1.3).

Example 3.3. *Let $A = (a^{ij})_{1\le i,j\le 2} = \operatorname{diag}(e^{x+y}, e^{x+y})$. One can easily check that $a^2_{x_1} = e^{x+y} > 0$. Then, by Theorem 1.2, there exists a d satisfing* (1.2) *and* (1.3).

Remark 3.1. *The sectional curvature condition provided in [3] works better than our Theorem 1.1 when a^{ij} is not of diagonal form. For instance, the Example 3.2 in [3].*

Acknowledgements

The author want to thank Professor Gengsheng Wang for pointing out the prototype of this result and his help on the examples.

References

[1] C. Bardos, G. Lebeau and J. Rauch, *Sharp sufficient conditions for the observation, control and stabilization of waves from the boundary, SIAM J. Control Optim.,* **30**(1992), 1024–1065.

[2] X. Fu, J. Yong and X. Zhang, *Exact controllability for multidimensional semilinear hyperbolic equations, SIAM J. Control Optim.,* **46**(2007), 1578–1614.

[3] P. F. Yao, *On the observability inequalities for exact controllability of wave equations with variable coefficients, SIAM J. Control Optim.,* **37**(1999), 1568–1599.

[4] P. F. Yao, *Modeling and Control in Vibrational and Structural Dynamics: A Differential Geometric Approach,* Chapman and Hall/CRC Press, Boca Raton, 2011.